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An achievable data-rate region subject to a stationary performance constraint for LTI plants

Eduardo I. Silva, Milan S. Derpich and Jan Østergaard

Abstract—This note studies the performance of control systems subject to average data-rate limits. We focus on a situation where a noisy LTI system has been designed assuming transparent feedback and, due to implementation constraints, a source coding scheme (with unity signal transfer function) has to be deployed in the feedback path. For this situation, and by focusing on a specific source coding scheme, we give a closed-form upper bound on the minimal average data-rate that allows one to attain a given performance level. Instrumental to our main result is the explicit solution of a related (and previously unsolved) signal-to-noise ratio minimization problem, subject to a closed loop performance constraint.

Index Terms—Networked control systems, average data-rate, signal-to-noise ratio, perfect reconstruction.

I. INTRODUCTION

NETWORKED control systems (NCSs) have recently received much attention in the literature (see, e.g., the papers in the special issue [1]). Within the NCS framework, a key issue is the characterization of the interplay between control objectives and communication constraints. This note focuses on NCSs subject to average data-rate constraints in the feedback path.

If stability is the only control objective, then the results in [2] provide a complete characterization of the minimum average data-rate that is compatible with the mean square stabilization of noisy LTI plants. This result establishes a key fundamental limitation when the problem of interest is mean-square stabilization. (We refer to [3] for a thorough discussion of this result.) On the other hand, when performance bounds subject to average data-rate constraints are sought, fewer results are available. It is known that, as the average data-rate approaches the minimum for stability, the performance becomes arbitrarily poor when disturbances are present [2], [3]. This holds irrespective of how the coder, decoder and controller are chosen. However, these results are based upon performance bounds that have not been shown to be tight in general [3].

An interesting performance-related result was presented in [4], where it is shown that, for noiseless LTI plants, one can essentially recover the best non-networked LQR performance with data-rates arbitrarily close to the minimum average data-rate for stabilization. Other results valid in the noiseless or bounded noise cases can be found in, e.g., [5]–[7]. The case

of unbounded support noise sources has been treated in [8] and [9], with an emphasis on stabilization and on achieving stationary state distributions.

Other relevant work dealing with unbounded support noise sources include [10] and [3]. Those works establish conditions for separation and certainty equivalence in the context of quadratic stochastic problems for fully observed plants, when data-rate constraints are present in the feedback path. Provided the encoder has a recursive structure, certainty equivalence and a *quasi*-separation principle hold [3]. Although this is an important result, [3] does not give a practical characterization of the optimal encoding policies. The results reported in [10] present a similar drawback. That work presents performance related results in terms of the sequential rate-distortion function, which is difficult to compute in general. Moreover, even for the cases where an expression for such function is available, it is not clear whether the sequential rate-distortion function is operationally tight [10, Section IV-C]. Other interesting work includes [11], where non-linear stochastic control problems over noisy channels are studied, but only implicit functional characterizations of the optimal control policies are presented.

In this note, we focus on a situation where a noisy LTI system has been designed assuming transparent feedback and, due to implementation constraints, a source coding scheme with unit signal transfer function has to be deployed in the feedback path. Such a situation arises, for example, when an LTI controller has been already designed for a given LTI plant without taking data-rate constraints into account. In such setting, one can aim at finding, at a second design stage, a source coding scheme so as to minimize the effects of communication constraints, while preserving (some of) the desired features of the non-networked controller design. We believe that, whilst conservative and suboptimal, such a design procedure has practical appeal.

The main contribution of this note is to provide, for the situation described above, a closed-form upper bound on the minimal average data-rate that allows one to attain a given performance level (as measured by the stationary variance of an error signal). To that end, we focus on the class of source coding scheme based on entropy coded dithered quantizers [12] presented in [13].¹ For the considered class of source coding schemes, average data-rate constraints can be enforced by imposing signal-to-noise ratio (SNR) constraints in a related control system that uses an additive noise channel.

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¹The work [13] does not study the interplay between average data-rates and closed loop performance.

This key result allows us to present our main contribution as an immediate consequence of the solution to a constrained SNR minimization problem. To our knowledge, our results are the first ones to provide a closed-form characterization of the interplay between average data-rates and performance when unbounded support noise sources are considered. Our work builds upon the results presented in [14], [15].

This note is organized as follows: Section II states the problem of interest. Section III studies the interplay between SNR constraints and performance for the considered NCS. These results are then used in Section IV to present our main result. Section V presents an example, and Section VI draws conclusions.

Notation: $\mathbb{E}\{\cdot\}$ denotes the expectation operator. If x is a random process, and provided the limit exists, $\sigma_x^2 \triangleq \lim_{k \rightarrow \infty} \mathbb{E}\{(x(k) - \mathbb{E}\{x(k)\})^T (x(k) - \mathbb{E}\{x(k)\})\}$. X^H denotes the conjugate transpose of the matrix X . \mathcal{RH}_2 is the set of all strictly proper and stable real rational transfer functions, and \mathcal{U}_∞ is the set of all stable, biproper and minimum phase real rational transfer functions. The usual norm in \mathcal{L}_2 is written $\|\cdot\|_2$ [16]. Unless otherwise stated, all signals and systems have arbitrary dimensions.

II. PROBLEM DEFINITION

A. Setup

Consider the NCS of Figure 1, where P is a proper real rational transfer function that models a given LTI system, d is a disturbance, e is a signal whose stationary variance σ_e^2 measures closed loop performance, y is a signal available for measurement, and u is an input of P . We partition P as

$$P \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad \begin{bmatrix} e \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} d \\ u \end{bmatrix}$$

and, for simplicity, we assume both P_{12} and P_{21} to be non-zero, and P_{22} to be SISO and strictly proper. The latter assumption implies that both y and u are scalar signals. However, d and e are allowed to have arbitrary dimensions. We also assume that the initial state of P , say x_o , is a second order random variable, and that d is a second order wide sense stationary process with spectral factor $\Omega_d \in \mathcal{U}_\infty$.

In our setup, P has been designed so as to achieve satisfactory performance when $u = y$. Accordingly, we assume that the feedback system of Figure 1 is internally stable and well-posed in the absence of communication constraints, i.e., when $u = y$. However, the feedback path comprises an error-free digital channel and thus the quantization (i.e., source encoding [17]) of y becomes mandatory. This task is carried out by an encoder that outputs the sequence of binary symbols s_c , which are then mapped into the input u by a decoder. Next section describes the class of encoder and decoders considered in this note.

B. The source coding scheme

We consider a standard feedback quantization scheme, where the digital channel input is generated by an *entropy coded dithered quantizer* (ECDQ) [12] that works associated

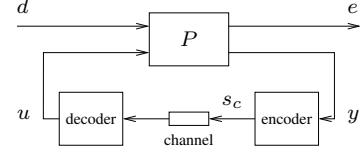


Fig. 1. NCS closed over a digital channel.

with causal LTI filters. At the encoder side, the channel input s_c (a binary word) is generated via

$$s_c(k) = \mathcal{H}_k(s(k), d_h(k)), \quad (1a)$$

$$s(k) = \mathcal{Q}(v(k) + d_h(k)), \quad (1b)$$

$$v = Ay - Fq, \quad (1c)$$

$$q = s - (v + d_h), \quad (1d)$$

where d_h is a real-valued random dither signal assumed to be available at both the encoder and decoder sides, $\mathcal{Q} : \mathbb{R} \rightarrow \mathcal{A} \triangleq \{i\Delta; i \in \mathbb{Z}\}$ corresponds to a uniform quantizer with step size $\Delta > 0$, $\mathcal{H}_k : \mathcal{A} \times \mathbb{R} \rightarrow \mathcal{A}_c(k)$ corresponds to the mapping describing an entropy-coder (EC; also called loss-less encoder [18, Ch.5]) whose output symbol is chosen according to the conditional distribution of $s(k)$, given $d_h(k)$, the set $\mathcal{A}_c(k)$ is, for every k , a set of prefix free binary words [18], and $A \in \mathcal{U}_\infty$ and $F \in \mathcal{RH}_2$ are given SISO LTI filters with deterministic initial states. On the other hand, the decoder output u is obtained via

$$u = A^{-1}(\hat{s} - d_h), \quad \hat{s}(k) = \mathcal{H}_k^{-1}(s_c(k), d_h(k)), \quad (2)$$

where $\mathcal{H}_k^{-1} : \mathcal{A}_c(k) \times \mathbb{R} \rightarrow \mathcal{A}$ corresponds to the mapping describing the entropy-decoder (ED) that is complementary to the EC at the encoder side, and all other symbols are as before. In (2), we implicitly used the fact that, since the feedback channel is error-free, s_c is available at the decoder side. It is also worth noting that, since EC-ED pairs are lossless [18],

$$\mathcal{H}_k^{-1}(\mathcal{H}_k(s(k), d_h(k)), d_h(k)) = s(k) \quad (3)$$

for every $s(k), d_h(k)$ and every $k \in \mathbb{N}_0$. We thus also have that $\hat{s} = s$.

Remark 1: The source coding scheme described above has a common source of randomness at both the sending and receiving ends: the dither. In principle, this requires the dither to be separately communicated to both channel ends. In practice, however, one can use synchronized pseudo-random number generators initialized with the same seeds. ■

For future reference, we denote by \mathbb{H} the set of all mapping sequences $\{(\mathcal{H}_k, \mathcal{H}_k^{-1}); k \in \mathbb{N}_0\}$, with \mathcal{H}_k and \mathcal{H}_k^{-1} described as above, and satisfying (3).

We denote the expected length of the symbol $s_c(k)$, measured in nats (1 nat equals $\ln 2$ bits), by $R(i)$ and define the average data-rate of the source coding scheme as

$$\mathcal{R} \triangleq \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} R(i).$$

The proposed source coding scheme has several key properties, as described below:

Theorem 1: Consider the NCS of Figure 1 where the encoder and decoder are described by (1) and (2), respectively.

Under the assumptions of Section II-A, and if $\Delta < \infty$ and the dither d_h is i.i.d., independent of (x_o, d) , and uniformly distributed on $(-\Delta/2, \Delta/2)$, then:

- (a) The quantization noise q in (1d) is i.i.d., independent of (x_o, d) , and uniformly distributed on $(-\Delta/2, \Delta/2)$.
- (b) The NCS is mean-square stable (MSS), i.e., the state of the system has a covariance bounded at all times that, in addition, converges as time grows unbounded.
- (c) There exists an EC-ED pair such that the average data-rate of the coding scheme satisfies

$$\mathcal{R} < \frac{1}{2} \ln(1 + \gamma) + \frac{1}{2} \ln\left(\frac{2\pi e}{12}\right) + \ln 2, \quad (4)$$

where

$$\gamma \triangleq \frac{\sigma_v^2}{\Delta^2/12}, \quad (5)$$

and σ_v^2 is the stationary variance of v in (1c). If, in addition, (x_o, d) is Gaussian, then $\mathcal{R} \geq \frac{1}{2} \ln(1 + \gamma)$ for any EC-ED pair.

Proof: Parts (a) and (c) are consequences of our assumptions, Theorems 1 and 2 in [12], Theorem 1 in [19], Corollary 5.5 in [15] and the fact that practical scalar EC-ED pairs achieve rates that are within $\ln 2$ from the absolute lower bound given by entropy [18, Ch.5] (see detailed discussion in [13], [15]). Part (b) readily follows from Part (a) and our assumptions. ■

By virtue of Part (a) of Theorem 1, and provided the dither d_h is properly chosen, the analysis of the NCS of Figure 1 is greatly simplified when the proposed coding scheme is used. Indeed, it immediately follows that

$$u = A^{-1}(q + v), \quad v = Ay - Fq, \quad (6)$$

where q is an i.i.d. sequence of uniformly distributed random variables, independent of both the external disturbance d and the initial plant state x_o . This implies that the transfer function from y to u is unity, and that quantization effects are introduced in an additive manner. In turn, Part (b) ensures that the MSS of the NCS is trivially guaranteed. Finally, Part (c) gives an upper bound on the average data-rate across the considered source coding scheme in terms of the stationary SNR γ (see (5)), and two additional terms. The first one appears because ECDQs generate a quantization noise that is uniformly distributed and not Gaussian [19], and the second one because practical EC-ED pairs are not perfectly efficient [18].

The discussion above shows that the proposed coding scheme allows one to address control problems subject to average data-rate limits by focusing on a simpler situation where additive noise and SNR constraints are present (see, e.g., [20], [21]). This stands in contrast to the literature reviewed in the introduction, and enables one to use well-known analysis and design tools to deal with average-data-rate-limited control systems.

Motivated by the above, we will next characterize the minimal stationary SNR γ that is needed to attain a given performance level, as measured by σ_e^2 . Then, in Section IV, we will use those results and (4) to immediately derive an upper

bound on the minimal average data-rate required to achieve the same performance level.

III. MINIMAL SNR SUBJECT TO A PERFORMANCE CONSTRAINT

The main result of this section is a closed-form expression for the minimal SNR γ that allows one to achieve a certain performance level D . Accordingly, we define, for $D \in \mathbb{R}^+$,

$$\gamma_D \triangleq \inf_{\substack{\Delta \in \mathbb{R}^+ \\ A \in \mathcal{U}_\infty, F \in \mathcal{RH}_2 \\ \sigma_e^2 \leq D}} \gamma. \quad (7)$$

Whilst an appropriate choice for the EC-ED pair is necessary for the average data-rate of the source coding scheme to satisfy (4), such choice does not influence γ nor σ_e^2 (see Part (a) of Theorem 1 and (6)), as long as the dither is properly chosen (which we henceforth assume is the case). Thus, consistent with the setup described in Section II, we consider in (7) that the quantization step Δ and the filters A and F are the only decision variables.

In order to characterize γ_D we define

$$S \triangleq (1 - P_{22})^{-1}, \quad T_{de} \triangleq P_{11} + P_{12}SP_{21}, \\ Y \triangleq |S|^2 \sqrt{P_{12}^H P_{12} P_{21} \Omega_d \Omega_d^H P_{21}^H}.$$

Theorem 2: Consider the setup and assumptions of Theorem 1. If $\|T_{de}\Omega_d\|_2^2 < D < \infty$, then

$$\gamma_D = h(\lambda_D) \\ \triangleq \exp\left(\frac{1}{\pi} \int_{-\pi}^{\pi} \ln\left(\sqrt{\frac{Y^2}{\lambda_D} + |S|^2} + \frac{Y}{\sqrt{\lambda_D}}\right) d\omega\right) - 1,$$

and $\lambda_D \in \mathbb{R}^+$ is the unique positive real satisfying

$$D = g(\lambda_D) \\ \triangleq \|T_{de}\Omega_d\|_2^2 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\lambda_D Y}{2\left(\sqrt{Y^2 + \lambda_D |S|^2} + Y\right)} d\omega.$$

On the contrary, if $D < \|T_{de}\Omega_d\|_2^2$ then the problem of finding γ_D is unfeasible, whereas achieving $D = \|T_{de}\Omega_d\|_2^2$ requires an infinite SNR γ .

Proof: Define $\sigma_q^2 \triangleq \mathbb{E}\{q(k)^2\} = \Delta^2/12$. Under our assumptions, it follows that

$$\gamma = \sigma_q^{-2} \|ASP_{21}\Omega_d\|_2^2 + \|1 - S + SF\|_2^2, \quad (8)$$

$$\sigma_e^2 = \|T_{de}\Omega_d\|_2^2 + \sigma_q^2 \|P_{12}SA^{-1}(1 - F)\|_2^2. \quad (9)$$

Using a contradiction based argument, it follows from (8) and (9) that the constraint $\sigma_e^2 \leq D$ is active at the optimum (note that this holds even if only A or F have to be chosen, and the remaining transfer function is fixed). Thus, (9) implies that at the optimum (i.e., when γ is minimized as in (7))

$$\sigma_q^2 = \frac{D - \|T_{de}\Omega_d\|_2^2}{\|P_{12}SA^{-1}(1 - F)\|_2^2}, \quad (10)$$

and also that (use (8) and the fact that $\|1 + X\|_2^2 = 1 + \|X\|_2^2$ for any $X \in \mathcal{RH}_2$)

$$\gamma = \frac{\|P_{12}SA^{-1}(1-F)\|_2^2 \|ASP_{21}\Omega_d\|_2^2}{D - \|T_{de}\Omega_d\|_2^2 + \|S(1-F)\|_2^2 - 1} \triangleq J_1(A). \quad (11)$$

It is also immediate to see from (9) that, if $D < \|T_{de}\Omega_d\|_2^2$, then the problem of interest is unfeasible. Also, (8) and the fact that our assumptions guarantee $P_{12}SA^{-1}(1-F) \neq 0$ and $ASP_{21}\Omega_d \neq 0$, imply that no finite γ yields $D = \|T_{de}\Omega_d\|_2^2$. We now proceed to prove our main claim and thus consider $D > \|T_{de}\Omega_d\|_2^2$. We will first assume that F is given, then that A is given and, at the final stage, we will use the results so obtained to prove the theorem.

- Consider a given $F \in \mathcal{RH}_2$ and recall the definition of $J_1(A)$ in (11). Disregarding the constraint $A \in \mathcal{U}_\infty$, the Cauchy-Schwartz inequality implies that J_1 is minimized by the idealized choice $A = A_F$, where

$$|A_F|^2 = |(1-F)| \sqrt{\frac{P_{12}^H P_{12}}{P_{21}\Omega_d\Omega_d^H P_{21}^H}}. \quad (12)$$

Thus, if $F \in \mathcal{RH}_2$ is given, then for γ to be arbitrarily close to its infimal value while satisfying $\sigma_e^2 \leq D$ (actually, while satisfying $\sigma_e^2 = D$), it is necessary to pick σ_q^2 as in (10) and $A \in \mathcal{U}_\infty$ such that $|J_1(A) - J_1(A_F)| < \epsilon_1$, for a sufficiently small $\epsilon_1 > 0$ (standard approximation results guarantee that this is always possible; see details in [22]).

- Consider a given $A \in \mathcal{U}_\infty$. We have from (11) that, at the optimum,

$$\gamma = \alpha_A^{-1} \|P_{12}SA^{-1}(1-F)\|_2^2 + \|S(1-F)\|_2^2 - 1 \triangleq J_2(F), \quad (13)$$

where

$$\alpha_A \triangleq \frac{D - \|T_{de}\Omega_d\|_2^2}{\|ASP_{21}\Omega_d\|_2^2}. \quad (14)$$

Define $\gamma_A \triangleq \inf_{F \in \mathcal{RH}_2} J_2(F)$ (see (13)). Lemma 1 in [23] and Lemma 10 on p. 171 in [16] immediately imply that

$$\gamma_A = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln\left(|S|^2 \left(1 + \frac{P_{12}^H P_{12}}{\alpha_A |A|^2}\right)\right) d\omega\right) - 1$$

and that the idealized optimal F , say F_A , satisfies

$$|1 - F_A|^2 = \frac{\gamma_A + 1}{|S|^2 \left(1 + \frac{P_{12}^H P_{12}}{\alpha_A |A|^2}\right)}. \quad (15)$$

Moreover, if σ_A^2 denotes the optimal value of σ_q^2 when A is given, then (10) and (14) yield

$$\sigma_A^2 = \frac{\alpha_A \|ASP_{21}\Omega_d\|_2^2}{\|P_{12}SA^{-1}(1-F_A)\|_2^2}. \quad (16)$$

We thus conclude that, if $A \in \mathcal{U}_\infty$ is given, then for γ to be arbitrarily close to its infimal value while satisfying

$\sigma_e^2 \leq D$ (actually, while satisfying $\sigma_e^2 = D$), it is necessary to pick $\sigma_q^2 = \sigma_A^2$ (see (16)) and an $F \in \mathcal{RH}_2$ such that $|J_2(F) - J_2(F_A)| < \epsilon_2$ for a sufficiently small $\epsilon_2 > 0$ (it follows from Lemma 1 in [23] and Lemma 10 on p. 171 in [16] that this is always possible).

- Now, we consider σ_q^2 and both A and F as design variables. At the joint optimum, the idealized optimal filters A_F and F_A in (12) and (15) must be reciprocally optimal. That is, (12) and (14)–(16) necessarily imply that the optimal noise variance σ_q^2 and the idealized jointly optimal filters, say σ_D^2 , A_D and F_D , satisfy

$$|A_D|^2 = |(1-F_D)| \sqrt{\frac{P_{12}^H P_{12}}{P_{21}\Omega_d\Omega_d^H P_{21}^H}}, \quad (17)$$

$$|1 - F_D|^2 = \frac{\gamma_D + 1}{|S|^2 \left(1 + \frac{P_{12}^H P_{12}}{\alpha_D |A_D|^2}\right)} \quad (18)$$

$$\sigma_D^2 = \frac{\alpha_D \|A_D S P_{21} \Omega_d\|_2^2}{\|P_{12} S A_D^{-1} (1 - F_D)\|_2^2} = \alpha_D, \quad (19)$$

where

$$\gamma_D = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln M d\omega\right) - 1, \quad (20)$$

$$M \triangleq |S|^2 \left(1 + \frac{P_{12}^H P_{12}}{\alpha_D |A_D|^2}\right), \quad (21)$$

$$\alpha_D \triangleq \frac{D - \|T_{de}\Omega_d\|_2^2}{\|A_D S P_{21} \Omega_d\|_2^2}. \quad (22)$$

Note that the last equality in (19) follows using (17) and (18). Also note that our assumptions, and the fact that $D > \|T_{de}\Omega_d\|_2^2$, guarantee that $\alpha_D \in \mathbb{R}^+$.

Solving (17) and (18) for $|1 - F_D|$ (and discarding a negative solution) we conclude that

$$|1 - F_D| = \frac{2(\gamma_D + 1)\alpha_D}{\sqrt{Y^2 + 4(\gamma_D + 1)\alpha_D^2 |S|^2} + Y}. \quad (23)$$

Define $\lambda_D \triangleq 4(\gamma_D + 1)\alpha_D^2$. Using (17) and (23) in (20) and (22) yields $\gamma_D = h(\lambda_D)$ with λ_D being such that $g(\lambda_D) = D$. Since $\alpha_D \in \mathbb{R}^+$, the first term on the right hand side of (20) is always finite and thus $\lambda_D \in \mathbb{R}^+$.

The uniqueness of λ_D and the fact that it always exists for any $D > \|T_{de}\Omega_d\|_2^2$ stems from the properties of h and g . Indeed, it can be easily shown that h is strictly decreasing, $\lim_{\lambda \rightarrow 0} h(\lambda) = \infty$, and $\lim_{\lambda \rightarrow \infty} h(\lambda) = \left(\prod_{i=1}^{n_p} |p_i|^2\right) - 1$, where p_i is the i^{th} unstable pole of P_{22} , whilst g is strictly increasing, $\lim_{\lambda \rightarrow 0} g(\lambda) = \|T_{de}\Omega_d\|_2^2$, and $\lim_{\lambda \rightarrow \infty} g(\lambda) = \infty$.

Since it is always possible to approximate A_D and F_D with $A \in \mathcal{U}_\infty$ and $F \in \mathcal{RH}_2$ to any desired level of accuracy (see details in [22]), the proof is thus completed. (The above suggests a procedure to find filters A , F and noise variance σ_q^2 (equivalently, quantization step Δ) so as to achieve $\gamma \leq \gamma_D + \epsilon_3$ while achieving $\sigma_e^2 \leq D$ (actually, while achieving $\sigma_e^2 = D$), $\forall \epsilon_3 > 0$: First, solve $g(\lambda) = D$ to obtain λ_D and thus γ_D and α_D . Choose $A \in \mathcal{U}_\infty$, $F \in \mathcal{RH}_2$ so as to approximate with sufficient

degree of accuracy the right hand sides of (17) and (23). Finally, choose $\sigma_q^2 = \sigma_D^2$ (see (19)). ■

Theorem 2 gives a closed-form expression for the minimal SNR that allows one to achieve a given performance level D in the NCS of Figure 1, under the constraint of having a unity transfer function from y to u . Our result is given in terms of the scalar parameter λ_D that satisfies $g(\lambda_D) = D$. Since g is monotone (see proof of Theorem 2), finding λ_D is a simple numerical problem that can be addressed using standard algorithms.

Remark 2: Theorem 2 connects the minimal SNR needed to achieve a given performance level, and the corresponding performance level, by means of two equations linked by the scalar parameter λ_D . This solution is akin to the “water filling equations” in rate-distortion theory [24], which relate optimal performance to the minimal data-rate needed to achieve it. ■

Remark 3: It follows from the proof of Theorem 2 that the minimal SNR compatible with MSS (i.e., the infimum of γ in (7) when the constraint on σ_e^2 is removed), say γ_∞ , is given by $\gamma_\infty = \left(\prod_{i=1}^{n_p} |p_i|^2\right) - 1$, where p_i is the i^{th} unstable pole of P_{22} . This provides an indirect proof of Theorem 9 in [13]. ■

Remark 4: Theorem 2 can be adapted to show that the minimal value of σ_e^2 that is achievable when the SNR γ is upper bounded by $\Gamma > \gamma_\infty$, say $[\sigma_e^2]_\Gamma$, is given by $[\sigma_e^2]_\Gamma = g(\lambda_\Gamma)$, where λ_Γ satisfies $h(\lambda_\Gamma) = \Gamma$. This observation solves the problem formulated in [23], without using the approximations made there, and extends [25] to more general architectures. ■

IV. AN UPPER BOUND ON THE MINIMAL AVERAGE DATA-RATE SUBJECT TO A PERFORMANCE CONSTRAINT

This section presents the main result of this note: A closed-form upper bound on the minimal average data-rate that guarantees a given performance level in the NCS of Figure 1.

We define, for $D \in \mathbb{R}^+$,

$$\mathcal{R}_D \triangleq \inf_{\substack{\Delta \in \mathbb{R}^+, \{(\mathcal{H}_k, \mathcal{H}_k^{-1}); k \in \mathbb{N}_0\} \in \mathbb{H} \\ A \in \mathcal{U}_\infty, F \in \mathcal{RH}_2 \\ \sigma_e^2 \leq D}} \mathcal{R}.$$

In contrast to the problem in (7), the EC-ED pair does play a role in this case (see Part (c) of Theorem 1).

Corollary 1: Consider the setup and assumptions of Theorem 1. If $\|T_{de}\Omega_d\|_2^2 < D < \infty$, then

$$\mathcal{R}_D < \frac{1}{2} \ln(1 + \gamma_D) + \frac{1}{2} \ln\left(\frac{2\pi e}{12}\right) + \ln 2, \quad (24)$$

where γ_D is characterized in Theorem 2. If, in addition, (x_o, d) is Gaussian, then $\mathcal{R}_D \geq \frac{1}{2} \ln(1 + \gamma_D)$. On the other hand, if $D < \|T_{de}\Omega_d\|_2^2$ then the problem of finding \mathcal{R}_D is unfeasible, whereas achieving $D = \|T_{de}\Omega_d\|_2^2$ requires an infinite average data-rate.

Proof: Immediate from Theorems 1 and 2. ■

Corollary 1 provides, for the NCS and source coding schemes considered in this note, a closed-form expression for an upper bound on the average data-rate that is required to attain a prescribed performance level. As discussed in the introduction, we believe that this results corresponds to

the first closed-form characterization (although given as an upper bound only) of the interplay between average data-rates and closed loop performance when unbounded support noise sources are considered.

In the Gaussian case, and within the class of considered source coding schemes, the bound provided by Corollary 1 is tight up to $\frac{1}{2} \ln\left(\frac{2\pi e}{12}\right) + \ln 2$ nats (1.254 bits) per sample. However, even in that case, our bound may be conservative when compared to the (still unknown) minimal average data-rate needed to attain a given performance level in the NCS of Figure 1, when causal but otherwise unconstrained source coding schemes are employed. Moreover, even within the class of source coding schemes based on dithered quantizers and LTI filters, our result is conservative since it assumes unity transfer function from y to u . We have dropped this assumption in [26]. However, no closed-form solutions seem to be computable in that case.

We end this section by noting that there exists a source coding scheme, within the proposed class, such that the average data-rate across it is guaranteed to satisfy the bound in (24):

Corollary 2: Consider the setup and assumptions of Theorem 1. Then, there exists a finite quantization step $\Delta > 0$, an EC-ED pair using Huffman coding [18], and filters A and F such that $\sigma_e^2 \leq D$ and $\mathcal{R} < \frac{1}{2} \ln(1 + \gamma_D) + \frac{1}{2} \ln\left(\frac{2\pi e}{12}\right) + \ln 2$.

Proof: Define $K \triangleq \frac{1}{2} \ln\left(\frac{2\pi e}{12}\right) + \ln 2$. The proof of Theorem 2 guarantees, $\forall \epsilon_3 > 0$, the existence of $A \in \mathcal{U}_\infty$, $F \in \mathcal{RH}_2$, and $\Delta > 0$ such that $\sigma_e^2 \leq D$ and $\gamma \leq \gamma_D + \epsilon_3$. Thus, there exist suitable A, F and Δ such that, $\forall \epsilon_4 > 0$, $\sigma_e^2 \leq D$ and $\frac{1}{2} \ln(1 + \gamma) < \frac{1}{2} \ln(1 + \gamma_D) + \epsilon_4$. For the same choices, Part (c) of Theorem 1 guarantees that there exists $\eta > 0$ such that $\mathcal{R} + \eta < \frac{1}{2} \ln(1 + \gamma) + K$ and, therefore, $\mathcal{R} < \frac{1}{2} \ln(1 + \gamma_D) + K + \epsilon_4 - \eta$. The result follows upon choosing ϵ_4 small enough so that $\eta > \epsilon_4$ holds. ■

Remark 5: Interestingly, if the EC-ED pair has memory, i.e., if it is allowed to exploit the complete history of the quantizer output symbols s and of the dither values d_h , then the bound given by Corollary 1 cannot be improved upon [13, Section V.B]. Thus, given our approach, the use of an EC-ED pair with memory does not allow one to establish better bounds on \mathcal{R}_D (see also remarks after Theorem 2 in [27]). ■

Remark 6: Using the observation made in Remark 3, it follows that the simple source coding scheme considered here allows one to achieve MSS in the NCS of Figure 1 at average data-rates that are guaranteed to be within $\frac{1}{2} \ln\left(\frac{2\pi e}{12}\right) + \ln 2$ nats (1.254 bits) per sample away from the absolute minimal rate for MSS identified in [2] (see [13] for additional discussions). ■

Remark 7: Using the results in Remark 4, it is possible to adapt Corollary 1 so as to characterize the best achievable performance subject to a given average data-rate constraint in the considered NCS. ■

V. AN EXAMPLE

Consider a SISO LTI plant with control input u , output y and input disturbance d , where

$$y = \frac{-0.75(z - 2)}{z^2 - 2z + 2}(u + d).$$

Assume that d is unit variance Gaussian white noise, and that the initial state of the plant is also Gaussian. This plant can be stabilized with (perfect) unity output feedback. We computed the right hand side of (24) for several values of $D > \|T_{de}\Omega_d\|_2^2 = 4.09$, and performed simulations using an actual ECDQ (we ran twenty 10^4 samples long realizations for each value of D). The ECDQ was simulated by adding uniform random numbers to v prior to quantization, and then subtracting them after decoding. However, the EC-ED pair worked conditioned upon only a (uniformly) quantized version of the true dither values. The results are presented in Figure 2.² In that figure, “Measured rate” corresponds to the average, over all realizations, of the number of bits per sample transmitted from encoder to decoder, and “Meas. cond. entropy of quantizer output” is an empirical estimate of the conditional entropy of the quantizer output s , given the quantized dither values.

The measured rates are well below the upper bound provided by Corollary 1. This suggests that the bound is rather loose, which is consistent with the fact that it is based upon worst case scenarios. Indeed, the gap between the measured rate (with conditioning) and the lower bound on \mathcal{R}_D presented in Figure 2 is about 0.49 bit per sample, which is smaller than the worst case gap given by $\frac{1}{2} \ln\left(\frac{2\pi e}{12}\right) + \ln 2$ nat per sample, i.e., 1.254 bit per sample. Since the measured conditional entropy of the quantizer output is about 0.26 bit per sample above the lower bound, it follows that the measured gap is composed of about 0.23 bit per sample due to the inefficiency of Huffman coding (considerably lower than the worst case rate loss of 1 bit per sample), and 0.26 bits per sample due to the fact that the quantization noise in the ECDQ is not Gaussian, but uniform [19], and due to the fact that the EC-ED pair works conditioned upon quantized dither values only.

As expected, the average data-rate required to achieve a performance arbitrarily close to the non-networked performance (i.e., $\sigma_e^2 = \|T_{de}\Omega_d\|_2^2 = 4.09$) grows unbounded. However, it is interesting to see that it suffices to use less than 6 (resp. 4) bits per sample (on average) to achieve a performance level that is within 1% (resp. 10%) from $\|T_{de}\Omega_d\|_2^2$.

Finally, we note that our bounds on \mathcal{R}_D converge rapidly when $D \rightarrow \infty$, which is also consistent with the behavior of the measured rates. Hence, the performance loss incurred when forcing the average data-rate to be lower than the upper bound provided by Corollary 1 when $D \rightarrow \infty$ is rather modest in this case.

VI. CONCLUSIONS

This note has studied a situation where an LTI system is designed assuming transparent feedback and, at a later design stage, a unity signal transfer function source coding scheme is to be utilized so as to minimize the effects of data-rate limits on closed loop performance. For this situation, we have explicitly characterized an upper bound on the minimal average data-rate that is needed to achieve a given performance level.

²In Figure 2, the number of quantized dither values is indicated in each case. As expected, conditioning the EC-ED pair on dither values, even though quantized, reduces the average data-rate.

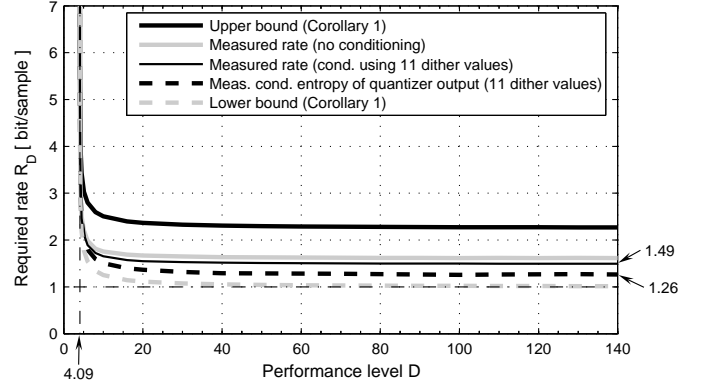


Fig. 2. Bounds on the minimal average data-rate required to attain a given performance level.

Future work should focus on extending the results of this note to cases where causal but otherwise unconstrained source coding schemes are used. The study of more general control architectures, and the joint design of controllers and source coding schemes, is part of another paper by the authors [26].

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